A Nonlinear $sl(2)$ Dynamics and New Quasiclassical Solutions for a Class of Quantum Coupled Systems

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Abstract

Hamiltonians of a wide-spread class of strongly coupled quantum system models are expressed as nonlinear functions of $sl(2)$ generators. It enables us to use the $sl(2)$ formalism, in particular, $sl(2)$ generalized coherent states (GCS) for solving both spectral and evolution tasks. In such a manner, using standard variational schemes with $sl(2)$ GCS as trial functions we find new analytical expressions for energy spectra and nonlinear evolution equations for cluster dynamics variables in mean-field approximations which are beyond quasi-harmonic ones obtained earlier. General results are illustrated on certain concrete models of quantum optics and laser physics.

PACS numbers: 03.70; 02.20

1 Introduction

For last decades a great attention has been paid to solve and to examine different dynamical problems for quantum strongly coupled systems whose interaction Hamiltonians are expressed by nonlinear functions of operators describing subsystems (see, e.g., [1-9] and references therein). However, as a rule, for these purposes numerical calculations are mainly used while analytical techniques available either deal with special forms of model Hamiltonians (including their different semiclassical versions) and initial quantum states [1-5,7-9] or require lengthy and tedious calculations (as it is the case, e.g., for the algebraic Bethe ansatz [6]).

Recently, a new universal Lie-algebraic approach has been developed [10-12] to get solutions of both spectral and evolution problems for some nonlinear quantum models of strongly coupled subsystems having symmetry groups G_{inv}. It was based on exploiting a formalism of polynomial Lie algebras g_{pd} as dynamic symmetry algebras g^{DS} of models under study with generators of these algebras g_{pd} being G_{inv}-invariant collective (cluster) dynamic variables in whose terms model dynamics are described completely. (In fact, such a reformulation of original problems in terms of g_{pd}-variables is similiar to the well-known procedure of exclusion of cyclic variables in classical mechanics [13].) Specifically, this approach enabled us to develop some efficient techniques for solving physical tasks in the case of $g^{DS} = sl_{pd}(2)$, when model Hamiltonians H are expressed as follows

$$H = aV_0 + gV_+ + g^*V_- + C, \quad [V_\alpha, C] = 0, \quad V_- = (V_+)^+, \qquad (1.1)$$

where $C = C([R_i])$ is a function of a set of commuting operators (model integrals of motion) $R_i, i = 1, 2, ...$ and $V_0, V_\pm$ are the $sl_{pd}(2)$ generators satisfying the commutation relations

$$[V_0, V_\pm] = \pm V_\pm, \quad [V_-, V_+] = \Psi(V_0 + 1) - \Psi(V_0), \quad \Psi(V_0) = A \prod_{i=1}^{n(\Psi)} (V_0 + \lambda_i(\{R_j\})), \qquad (1.2a)$$

$$[\Psi(R_0), V_\alpha] = 0 \quad \forall \alpha = 0, \pm, \qquad \Psi(R_0) = \Psi(V_0) - V_+V_-, \tag{1.2b}$$

where $n(\Psi)$ is the polynomial Ψ degree in the variable V_0, $\Psi(R_0)$ is the $sl_{pd}(2)$ Casimir operator (with R_0 being the "lowest weight operator") and hereafter the identity operator symbol I is omitted in expressions like $\Psi(V_0 + \alpha I)$. The structure polynomials $\Psi(V_0)$ depend additionally on $\{R_i, i = 1, \ldots\}$, and their exact expressions for some wide-spread classes of concrete models were given in [10-12] (see also Section 5).

All techniques [10-12] essentially use expansions of evolution operators $U_H(t)$, generalized coherent states (GCS), energy eigenfunctions $|E_f>$ and other important physical quantities by power series in the $sl_{pd}(2)$ shift generators $V_\pm$ as well as commutation relations (1.2) and the characteristic equation

$$(V_+V_- - \Psi(V_0) \equiv -\Psi(R_0))|_{L(H)} = 0 \tag{1.3}$$

fulfilled on Hilbert spaces $L(H)$ of quantum model states due to the complementarity of groups G_{inv} and algebras $sl_{pd}(2)$ [10]. Specifically, Eq. (1.3) implies a spectral decomposition

$$L(H) = \sum_{[l_i]} L([l_i]), \qquad L([l_i]) \equiv Span\{|[l_i]; v\rangle = \mathcal{N}([l_i]; v)V_+^v|[l_i]\rangle\},$$

$$V_0|[l_i]; v\rangle = (l_0 + v)|[l_i]; v\rangle, \quad R_i|[l_i]; v\rangle = l_i|[l_i]; v\rangle, i = 0, 1, ..., \quad V_-|[l_i]\rangle = 0 \tag{1.4}$$

of spaces $L(H)$ in direct sums of the subspaces $L([l_i]) \equiv L(l_0, l_1, \ldots)$ which are irreducible with respect to joint actions of algebras $sl_{pd}(2)$ and symmetry groups G_{inv}; lowest weights l_0 depend on other quantum numbers $l_i, i = 1, \ldots$ due to the relation $\Psi(l_0) = 0$ implied by Eq. (1.3). From the physical point of view, the decomposition (1.4) means that the model Hamiltonian matrices in the symmetry adapted orthonormalized bases $\{|[l_i]; v\rangle\}$ have block-diagonal forms and subspaces $L([l_i])$ describe specific "$sl_{pd}(2)$-domains" evolving independently in time under action of Hamiltonians (1.1). We also distinguish compact $(su_{pd}(2))$ and non-compact $(su_{pd}(1,1))$ versions of $sl_{pd}(2)$ algebras depending on whether dimensions $d([l_i])$ of the spaces $L([l_i])$ are finite or infinite.

Then, using restrictions $H_{[l_i]}$ of Eq. (1.1) on $L([l_i])$, simple algebraic calculation schemes were developed for finding evolution operators $U_H(t) = \sum_{f=-\infty}^{\infty} V_+^f \ u_f(V_0; t)$, amplitudes $Q_v(E_f) = \langle [l_i]; v|E_f\rangle$ of energy eigenstates $|E_f\rangle$ expansions in orthonormalized bases $\{|[l_i]; v\rangle\}$) and appropriate energy spectra $\{E_f\}$ of bound states [10]. In the paper [11] some explicit integral representations were found for amplitudes $Q_v(E)$, eigenenergies $\{E_a\}$ and "evolution coefficients" $u_f(V_0; t)$ with the help of a specific "dressing" (mapping) of solutions of some auxiliary exactly solvable tasks with the dynamic algebra $sl(2)$.

However, all exact results obtained do not yield simple working formulas for analysing models (1.1) and revealing different physical effects (e.g., a structure of collapses and revivals of the Rabi oscillations [2,8], bifurcations and singularities of quasiclassical solutions [5] etc.) at arbitrary initial quantum states of models. Therefore, it is necessary to develop some simple techniques, in particular, to get some closed, perhaps, approximate expressions for evolution operators, energy eigenvalues and wave eigenfunctions, which would describe main physical peculiarities of model dynamics with a good accuracy (cf. [5,8,9]). Below we examine some possibilities along these lines for models (1.1)-(1.4) by means of reformulating them in terms of the usual $sl(2)$ algebra formalism and developing variational schemes corresponding to quasiclassical approximations (QAs) for these models by analogy with developments [5,14-16].

The work is organized as follows. In Section 2 we first reformulate models (1.1)-(1.4) in terms of the usual Lie algebra $sl(2)$ formalism, and then discuss possibilities of extending the standard $sl(2)$-techniques to analyse such reformulated models. In Section 3 a scheme is given for obtaining QAs of these models by using variational principles [5,17] and energy functionals constructed with the help of the $SL(2)$ group GCS [16]; these QAs are new for original models because they take into account a strong coupling of interacting subsystems in contrast with standard QAs. Specifically, in such a manner new analytical expressions are obtained for energy spectra which are essentially non-equidistant on each subspace $L([l_i])$ with its dimension $d([l_i]) \geq 4$. In Section 4 we discuss such approximations for a quasiclassical description of dynamics of $sl(2)$-clusters (characteristic model exicitations) and time evolution of uncoupled dynamical variables; specifically, nonlinear evolution equations of the Bloch type are obtained for $sl(2)$-cluster variables. In Section 5 a specification of general results is given for a class of models widely used in quantum optics and laser physics. In conclusion some of prospects of developing this approach are discussed.

2 A nonlinear $sl(2)$ formulation and a general operator analysis of quantum models with linear $sl_{pd}(2)$ dynamic algebras

We can reformulate models (1.1), (1.4) in terms of $sl(2)$ generators using a realization of the $sl_{pd}(2)$ algebras in terms of special elements of extended enveloping algebras $\mathcal{U}_\Psi(sl(2))$ of the familiar algebra $sl(2)$ [12]. This realization is established via the generalized Holstein-Primakoff mapping [10]

$$Y_0 = V_0 - R_0 \mp \hat{J}, \ Y_+ = V_+[\Phi(Y_0)]^{-1/2}, \ \Phi(Y_0) = \frac{\Psi(Y_0 + R_0 \pm \hat{J} + 1)}{(\hat{J} \mp Y_0)(\pm \hat{J} + 1 + Y_0)}, \ Y_- = (Y_+)^+, \ (2.1a)$$

$$[Y_0, Y_\pm] = \pm Y_\pm, \quad [Y_-, Y_+] = \mp 2Y_0 \qquad (2.1b)$$

where Y_α are the $sl(2)$ generators, $R_0, \mp \hat{J}$ are lowest weight operators of the $sl_{pd}(2)$ and $sl(2)$ algebras respectively: $\Psi(R_0))|_{L(H)} = 0, [\hat{J}(\hat{J} \pm 1) \mp Y_+Y_- - Y_0^{(2)}]|_{L(H)} = 0$. Appropriate specifications of Eqs. (2.1a) on subspaces $L([l_i])$ are obtained by the substitution $R_0 \to l_0, \hat{J} \to J$ and hereafter upper/lower signs in (2.1) corresponding to the $su(2)/su(1,1)$ algebras are chosen for finite/infinite dimensions $d([l_i])$ of the spaces $L([l_i])$. Note that, by definition (2.1), functions $\Phi(Y_0)$ on $L([l_i])$ are polynomials of the $(n(\Psi) - 2)$-th degree in the variable Y_0 at relevant values of J [10,12].

Then, using Eqs. (2.1) one may re-write Hamiltonians (1.1) in terms of Y_α as follows,

$$H = aY_0 + Y_+g(Y_0) + g^+(Y_0)Y_- + C',$$

$$g(Y_0) = g\sqrt{\Phi(Y_0)}, \qquad C' \equiv C'([R_i], \hat{J}) = C([R_i]) + a(R_0 \pm \hat{J}) \qquad (2.2)$$

Restrictions $H_{[l_i]} \equiv P_{[l_i]}H$ of Hamiltonians (1.1) on spaces $L([l_i])$ (with $P_{[l_i]} = \sum_v |[l_i]; v\rangle\langle v; [l_i]|$ being appropriate central projectors) are obtained by the substitution $R_0 \to l_0, \hat{J} \to J$ in Eq.(2.2). Respectively, basis vectors $|[l_i]; v\rangle$ of spaces $L([l_i])$ are given in terms of Y_α as follows,

$$|[l_i]; v\rangle = \mathcal{N}(J, v)(Y_+)^v|[l_i]\rangle \qquad (2.3)$$

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where $\mathcal{N}^{-2}(J,v) = v!(2J)!/(2J-v)!$ for $su(2)$ and $\mathcal{N}^{-2}(J,v) = v!\Gamma(2J+v)/\Gamma(2J)$ for $su(1,1)$. Evidently, Eq. (2.2) resembles Hamiltonians of semi-classical $sl(2)$ "linearized" versions of matter-radiation interaction models [4,8,9,12] but with operator (intensity-dependent) coupling coefficients $g(Y_0)$ (cf. [3,4,7]). Emphasize, however, a collective (not associated with a single subsystem) nature of operators Y_α that leads, when substituting $g(Y_0)$ in Eq. (2.2) by an "effective coupling constant", to a non-standard ("cluster") QA of original models [10] distinguished from standard semi-classical limits [8,18,19] where a part of interacting subsystems is described classically.

If $n(\Psi) = 2$, then $\Phi(Y_0) = 1, sl_{pd}(2) = sl(2), R_0 = \mp\hat{J}$, and we have a powerful tool for solving both spectral and evolution tasks yielded by the GCS formalism [16] related to the $SL(2)$ group displacement operators

$$S_Y(\xi = re^{i\theta}) = \exp(\xi Y_+ - \xi^* Y_-) = \exp[t(r)e^{i\theta}Y_+]\exp[-2\ln c(r)Y_0]\exp[-t(r)e^{-i\theta}Y_-] =$$

$$\sum_{f=-\infty}^{\infty} Y_+^f S_f^Y(Y_0;\xi), \quad Y_+^{-k} \equiv Y_-^k\left([\Psi_2(Y_0)]^{(k)}\right)^{-1} \text{ for } k > 0 \qquad (2.4a)$$

where $t(r) = \tan r/\tanh r, c(r) = \cos r/\cosh r, s(r) = \sin r/\sinh r$ for $su(2)/su(1,1)$ and

$$[\Psi_2(Y_0)]^{(k)} \equiv (\pm 1)^k(\pm\hat{J} + Y_0)^{(k)}(\pm\hat{J} - Y_0 + k)^{(k)}, \quad A^{(x)} \equiv A(A-1)...(A-x+1) \qquad (2.4b)$$

$$S_f^Y(Y_0;\xi) = \frac{(e^{i\theta}t(r))^f}{f!} {}_2F_1(\mp J - Y_0, -Y_0 \pm \hat{J} + 1; f+1; \pm[s(r)]^2)\exp[-2\ln c(r)Y_0] \qquad (2.4c)$$

with ${}_2F_1(...)$ being the Gauss hypergeometric function [20].

Specifically, in this case, using the well-known $sl(2)$ transformation properties of operators Y_α under the action of $S_Y(\xi)$ [16,12],

$$S_Y(\xi)Y_+S_Y(\xi)^\dagger \equiv Y_+(\xi) = [c(r)]^2 Y_+ \pm e^{-i\theta}[s(2r)Y_0 - e^{-i\theta}[s(r)]^2 Y_-],$$

$$S_Y(\xi)Y_0 S_Y(\xi)^\dagger \equiv Y_0(\xi) = c(2r)Y_0 - \frac{s(2r)}{2}[e^{i\theta}Y_+ + e^{-i\theta}Y_-, \quad Y_-(\xi) = (Y_+(\xi))^\dagger, \qquad (2.5)$$

Hamiltonians H can be transformed into the form

$$\tilde{H}(\xi) = S_Y(\xi)HS_Y(\xi)^\dagger = C' + Y_0 A_0(a,g;\xi) + Y_+ A_+(a,g;\xi) + Y_- A_+^*(a,g;\xi) \qquad (2.6a)$$

At the values $\xi_0 = \frac{g}{|g|}r$ of the parameter ξ with $\tan 2r/\tanh 2r = \frac{2|g|}{a}$ for $su(2)/su(1,1)$ one gets $A_+(a,g;\xi) = 0$, and the Hamiltonian $\tilde{H}_{[l_i]}(\xi)$ takes the form

$$\tilde{H}(\xi_0) = C' + Y_0\sqrt{a^2 \pm 4|g|^2} \qquad (2.6b)$$

which is diagonal on eigenfunctions $|[l_i];v\rangle$. Therefore, original Hamiltonians H have within each $L([l_i])$ equidistant spectra with eigenenergies

$$E([l_i];v) = \tilde{C} + (\mp J + v)\sqrt{a^2 \pm 4|g|^2}, \quad \tilde{C} = C'([l_i];J) \qquad (2.7a)$$

and eigenfunctions

$$|[l_i];v;\xi_0\rangle = S_Y(\xi_0)^\dagger|[l_i];v\rangle = \exp(-\xi_0 Y_+ + \xi_0^* Y_-)|[l_i];v;\rangle = \sum_{f\geq 0} S_{fv}(J;g,r)|[l_i];f\rangle,$$

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$$S_{fv}(J;g,r) = \frac{[c(r)]^{2(\pm J-v)}(-\frac{g}{|g|}t(r))^{f-v}\mathcal{N}(J,v)}{(f-v)!\mathcal{N}(J,f)} {}_2F_1(-v,-v\pm 2J+1;f-v+1;\pm[s(r)]^2),$$

$$(2.7b)$$

where $t(r) = \pm\left(-a+\sqrt{a^2\pm 4|g|^2}\,\right)/2|g|$ and $\mathcal{N}(J,...)$ are normalization constants from Eq. (2.3).

Similarly, if $sl_{pd}(2) = sl(2)$, operators $S_Y(\xi(t))$ are "principal" parts in the evolution operators $U_H(t) = \exp(i\alpha(t)Y_0)S_Y(\xi(t))$ with $\alpha(t),\xi(t)$ being c-number functions in t which are determined from disentangling the exponent $\exp(\frac{it}{\hbar}H)$ (or, when g are time-dependent functions, from a set of non-linear differential equations corresponding to classical motions) [16,19].

However, for arbitrary degrees n of polynomials $\Psi(V_0)$ Hamiltonians (2.2) are essentially nonlinear in $sl(2)$ generators Y_α, and, therefore, the situation is very changed. Specifically, in general cases it is unlikely to diagonalize H with the help of operators $S_Y(\xi)$ since analogs of Eq. (2.6a),

$$\tilde{H}(\xi) = S_Y(\xi)HS_Y(\xi)^\dagger = aY_0(\xi) + Y_+(\xi)g(Y_0(\xi)) + g^+(Y_0(\xi))Y_-(\xi) + C', \qquad (2.8)$$

and even their restrictions $\tilde{H}_{[l_i]}(\xi) = S_Y(\xi)H_{[l_i]}S_Y(\xi)^\dagger$ on multi-dimensional spaces $L([l_i])$ contain (after expanding $g(Y_0(\xi))$ in power series) many terms with higher powers of $Y_\pm$. The task is also not simplified when using in Eq. (2.8) operators $S_V(\xi) = \exp(\xi V_+ - \xi^*V_-)$ instead of $S_Y(\xi)$ because we have not suitable analogs of the "disentangling theorem" (2.4) and finite-dimensional transformations (2.5) for operators $S_V(\xi)$ [12]. Therefore it is necessary to use in Eq. (2.8) more general (perhaps, non- or multi-parametric) forms of diagonalizing operators S given, e.g., by power series

$$S = \sum_{f=-\infty}^{\infty} Y_+^f S_f(Y_0), \quad Y_+^{-k} \equiv Y_-^k\left([\Psi_2(Y_0)]^{(k)}\right)^{-1} \ \forall\, k > 0 \qquad (2.9)$$

with undetermined (unlike Eq. (2.4)) coefficients $S_f(Y_0)$ and satisfying the unitarity conditions $SS^\dagger = S^\dagger S = I$.

Substituting Eq. (2.9) in the scheme (2.8) one gets after some algebra nonlinear analogs of Eqs. (2.6a)

$$\tilde{H} = SHS^\dagger = C' + \sum_{f=-\infty}^{\infty} Y_+^f \tilde{h}_f(Y_0) \qquad (2.10a)$$

where

$$\tilde{h}_f(Y_0) = \sum_{k=-\infty}^{\infty} [\Psi_2(Y_0)]^{(k)} S_k^*(Y_0 - k)[a(Y_0 - k)S_{k+f}(Y_0 - k)+$$

$$g\sqrt{\Phi(Y_0-k)}S_{k-1+f}(Y_0-k+1) + g^*\sqrt{\Phi(Y_0-k-1)}S_{k+1+f}(Y_0-k-1)\Psi_2(Y_0-k)] \quad (2.10b)$$

and $\tilde{h}_{-f}(Y_0) = \tilde{h}_f^*(Y_0 - f)[\Psi_2(Y_0)]^{(f)}$, $[\Psi_2(Y_0)]^{(-f)} \equiv ([\Psi_2(Y_0 + f)]^{(f)})^{-1}$ for all $f > 0$.

The conditions $\tilde{h}_f(Y_0) = 0$ for all $f \neq 0$ yield nonlinear analogs of Eqs. (2.6b), (2.7b),

$$\tilde{H}^0 = S^0 H S^{0\dagger} = C' + \tilde{h}_0^0(Y_0) = \sum_{k=-\infty}^{\infty} [\Psi_2(Y_0)]^{(k)} S_k^{0*}(Y_0 - k)[a(Y_0 - k)S_k^0(Y_0 - k)+$$

$$g\sqrt{\Phi(Y_0-k)}S_{k-1}^0(Y_0-k+1) + g^*\sqrt{\Phi(Y_0-k-1)}S_{k+1}^0(Y_0-k-1)\Psi_2(Y_0-k)], \quad (2.11a)$$

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$$E([l_i]; v) = \tilde{C} + \langle[l_i]; v|\tilde{h}_0^0(Y_0)|[l_i]; v\rangle = \tilde{C} + \tilde{h}_0^0(\mp J + v) \qquad (2.11b)$$

expressed in terms of the coefficients $S_f^0(Y_0)$ which satisfy Eqs. (2.10b) with $\tilde{h}_f(Y_0) = 0$ for $f \neq 0$ and simultaneuosly are solutions of the set of algebraic operator equations,

$$[-aY_0 + \tilde{h}_0^0(Y_0 + f)]S_f^0(Y_0) =$$

$$g\sqrt{\Phi(Y_0)}S_{f-1}^0(Y_0+1) + g^*\sqrt{\Phi(Y_0-1)}S_{f+1}^0(Y_0-1)\Psi_2(Y_0) \quad \forall |f| > 1 \qquad (2.12)$$

resulting from the condition $S^0 H = [C' + \tilde{h}_0^0(Y_0)]S^0$ [12] where S^0 is given by Eq. (2.9) with "coefficients" $S_f^0(Y_0)$.

In the case of $\Phi(Y_0) = 1$ Eqs. (2.12) are solved in terms of hypergeometric functions $_2F_1(...)$ as it follows from Eq. (2.4c), but in general they, probably, determine certain q-special functions due to relations of $sl_{pd}(2)$ algebras with certain q-deformed algebras [21]. Without using any specifications of operators S, due to the relation $Q_v(E_f) = S_{f-v}^*(\pm J + v)\frac{\mathcal{N}(J,v)}{\mathcal{N}(J,f)}$, the task of solving these equations is equivalent to that for finding amplitudes $Q_v(E_f)$ related to new classes of orthogonal functioms [10]. Note that this task is simplified in the compact $(su(2))$ case, when all subspaces $L([l_i])$ have finite dimensions $d([l_i]) = 2J + 1$, and all series in Eqs. (2.9)-(2.11) are terminating due to Eq. (1.3) and the relation $(Y_\pm)^{2J+1}|_{L([l_i])} = 0$. Therefore, eigenfunctions $|E_v\rangle = S^\dagger|[l_i]; v\rangle$ may be represented by polynomials

$$|E_v\rangle = \sum_{f=0}^{2J} Q_f^v Y_+^f|[l_i]\rangle = A_v \prod_r (Y_+ - \kappa_r^v)|[l_i]\rangle, \qquad (2.13)$$

where amplitudes Q_f^v are expressed as symmetric functions in variables κ_r^v:

$$Q_{2J}^v = A_v, \qquad Q_{2J-1}^v = -A_v \sum_{r=0}^{2J} \kappa_r^v, ...,$$

$$Q_{2J-f}^v = (-1)^f A_v \sum_{1 \leq r_1 < r_2 < ... < r_f \leq 2J} \kappa_{r_1}^v \kappa_{r_2}^v ... \kappa_{r_f}^v, ..., \quad Q_0^v = (-1)^{2J} A_v \kappa_{r_1}^v \kappa_{r_2}^v ... \kappa_{r_{2J}}^v, \qquad (2.14)$$

At the same time eigenenergies $E([l_i]; v)$ determined by the boundary condition [10]

$$[(l_0 + 2J)a - E([l_i]; v) + C([l_i])]Q_{2J}^v + gQ_{2J-1}^v = 0, \qquad (2.15)$$

can be written down in the form

$$E([l_i]; v) = \tilde{C} + Ja - g\sum_{r=0}^{2J} \kappa_r^v, \quad \tilde{C} = C([l_i]) + (l_0 + J)a \qquad (2.16)$$

of a sum of $2J + 1$ spectral functions as it is prescribed by the algebraic Bethe ansatz [6]. In fact, Eqs. (2.13)-(2.16) give for models given by Eqs. (1.1)-(1.4) a new, G_{inv}-invariant formulation of this ansatz in terms of the $su(2)$ algebra which is simpler and more efficient in comparison with its initial non-invariant version [6] because the algorithm [10] for finding amplitudes Q_f^v and eigenenergies $E([l_i]; v)$ does not require a preliminary determination of parameters κ_r^v. We also note that in the resonance case (when $a = 0$ in (2.2)), using Eqs. (2.12), one can get analytical solutions for amplitudes Q_f^v in the form of multiple sums which, however, are not suitable for practical purposes.

So, direct generalizations of "linear" schemes (2.6) to the case of non-linear Hamiltonians (2.2) do not yield simple analytical formulas for exact solutions of spectral tasks; a similar situation is also with respect to evolution problems. Nevertheless, the formalism of the $SL(2)$ GCS $|[l_i]; v; \xi\rangle = S_Y(\xi)^\dagger |[l_i]; v\rangle$ can be an efficient tool for analysing such models [5,10,14-16] and for getting approximate analytical solutions of both spectral and evolution problems. Specifically, a simplest example of such approximations was given in [10] by mapping (with the help of the change $V_\alpha \to Y_\alpha$) Hamiltonians (1.1) into Hamiltonians $H_{sl(2)}$ which are linear in $sl(2)$ generators Y_α (but with modified constants $\tilde{a}, \tilde{g}$) and have on each fixed subspace $L([l_i])$ equidistant energy spectra given by Eq. (2.7a). However, this (quasi)equidistant approximation, in fact, corresponding to a substitution of certain effective coupling constants $\tilde{g}$ instead of true operator entities $g(Y_0)$ in Eq. (2.2), does not enable to display many peculiarities of models (1.1) related to essentially non-equidistant parts of their spectra. Therefore, it is needed in corrections, e.g., with the help of iterative schemes [8,14,15]; specifically, one may develop perturbative schemes by using expansions of operator entities $g(Y_0)$ in Taylor series in Y_0 as it was made implicitly for the Dicke model in [8,9]. But there exists a more effective, incorporating many peculiarities of models (1.1), way to amend the quasi-equidistant approximation.

3 $SL(2)$ quasiclassical approximations: energy functionals and variational energy spectra

This way is in applying $SL(2)$ GCS $|[l_i]; v; \xi\rangle$ from Eq. (2.7b) as trial functions in the variational schemes [17] of determining energy spectra and quasiclassical dynamics [5,15]. Indeed, because of the isomorhism of quantum and quasiclassical dynamics for $sl(2)$ linear Hamiltonians [15,16], the results (2.7) can be obtained with the help of the variational scheme determined by the stationarity conditions

$$a) \quad \frac{\partial \mathcal{H}([l_i]; v; \xi)}{\partial \theta} = 0, \qquad b) \quad \frac{\partial \mathcal{H}([l_i]; v; \xi)}{\partial r} = 0 \tag{3.1}$$

for the energy functional $\mathcal{H}([l_i]; v; \xi) = \langle [l_i]; v; \xi | H | [l_i]; v; \xi \rangle = \langle [l_i]; v | C' + Y_0 A_0(a, g; \xi) + Y_+ A_+(a, g; \xi) + Y_- A_+^*(a, g; \xi) | [l_i]; v \rangle$ (cf. (2.6a)). Similarly, following the standard variational approach [17,5], the calculation schemes (3.1) may be extended to the case of nonlinear Hamiltonians (2.2) by using the energy functional

$$\mathcal{H}^{cq}([l_i]; v; \xi) = \langle [l_i]; v; \xi | H | [l_i]; v; \xi \rangle = \langle v; [l_i] | \tilde{H}(\xi) | [l_i]; v \rangle \tag{3.2}$$

where superscript cq denotes "cluster" (strongly correlated) QAs (as contrasted with standard QAs dealing with weakly or non-correlated subsystems) and $\tilde{H}(\xi)$ are given by Eq. (2.8) or Eqs. (2.10) with $S_f(Y_0) = S_f^Y(Y_0; \xi)$ from Eqs. (2.4c).

Note that, due to the Hermitian conjugacy relations $Y_+ g(Y_0) = (g^+(Y_0) Y_-)^\dagger$, the condition (3.1a) gives $e^{i\theta} = g/|g|$ (as in the linear case); in fact, this condition (3.1a) can be eliminated at once by the simple gauge transformation $Y_\alpha \to \exp(-i\alpha\gamma) Y_\alpha, g = |g| \exp(i\gamma)$, in Eqs.(2.2) which preserves commutation relations (2.1b). Furthermore, due to the form of trial functions and the unitarity of operators $S_Y(\xi)$, it is sufficiently to solve Eq. (3.1b) only for finding states $|[l_i]; v = 0; \xi\rangle$ and to use the only real root of one of Eqs. (3.1b) for all v that ensures

automatically the orthogonality of eigenfunctions. Naturally, results thus obtained are not expected to coincide with exact solutions on all subspaces $L([l_i])$ due to an essential non-linearity of Hamiltonians (2.2b) and their non-equivalence (unlike Eq. (2.6b)) to diagonal parts of Eq. (2.10a); however, they yield "smooth" (analytical) solutions which are in a sense most close to exact ones (cf. [5,14]). Without discussing all aspects of such extensions we give below two approximations for energy spectra obtained by inserting in Eq. (3.2) $\tilde{H}(\xi)$ given by Eqs. (2.10) and Eq. (2.8) respectively.

In the first case, using Eq. (2.5) for $Y_0(\xi_0)$, Eq. (2.7b) for $|[l_i]; v; \xi\rangle$ and defining relations for the $sl(2)$ algebra, one gets the following "cluster" QAs

$$E^{cq}([l_i]; v) = \tilde{C} + a(v \mp J)c(2r) + \Re\{g\langle [l_i]; v; \xi_0|Y_+\sqrt{\Phi(Y_0)}|[l_i]; v; \xi_0\rangle\} =$$

$$\tilde{C} + a(v \mp J)c(2r) - 2|g|\sum_{f \geq 0}|S_{fv}(J; g, r)S_{f+1v}(J; g, r)|\sqrt{\Psi(l_0 + 1 + f)}, \qquad (3.3a)$$

$$E^{cq}([l_i]; 0) = \tilde{C} \mp aJc(2r) - 2|g|[c(r)]^{\pm 4J}\sum_{f \geq 0}\frac{\sqrt{\Phi(\mp J + f)}(t(r))^{2f+1}}{f!(f+1)!\mathcal{N}^2(J, f+1)} \qquad (3.3b)$$

for energy eigenvalues $E^{cq}([l_i]; v) = \mathcal{H}^{cq}([l_i]; v; \xi_0)$ where $\tilde{C} = C([l_i]) + a(l_0 \pm J)$, $\xi_0 = rg/|g|$, $2\Re\{A\} = A^* + A, c(r) = \cos r/\cosh r$, $\Psi(l_0 + 1 + f) = \Phi(\mp J + f)(2J \mp f)(f + 1)$, $\mathcal{N}(J, ...)$ are normalization constants from Eq. (2.3) and functions $S_{fv}(J; g, r)$ are given by Eq. (2.7b) but with values of the parameter r determined by real solutions of the algebraic equation

$$\frac{2aJ}{|g|}\alpha(1 \pm \alpha^2)^{\pm 2J-1} = \sum_{f \geq 0}\frac{\sqrt{\Phi(\mp J + f)}(t(r))^{2f}}{f!(f+1)!N^2(J, f+1)}[\mp 4\alpha^2 j + (1 \pm \alpha^2)(2f + 1)],$$

$$\alpha = t(r) = \tan r/\tanh r \qquad (3.4)$$

which follows from Eqs. (3.1b) and (3.3b). Obviously, unlike the linear case, diagonalizing values of r depend on both constants g, a and quantum numbers l_i labeling G-invariant subspaces $L([l_i])$.

In the second case it is difficult to obtain exact analytical formulas like Eq. (3.3) due to presence of square roots in Eq. (2.8). However we can get another approximation for energy spectra if replacing the energy functionals (3.2) by their (corresponding to the Ehrenfest theorem with respect to cluster variables Y_i) mean-field approximations

$$\mathcal{H}^{cmf}([l_i]; v; \xi) = a < Y_0(\xi) > +2\Re\{< Y_+(\xi) > \tilde{g}(< Y_0(\xi) >)\} + \tilde{C},$$

$$< Y_\alpha(\xi) >= \langle v; [l_i]|Y_\alpha(\xi)|[l_i]; v\rangle, \quad 2\Re\{A\} = A + A^* \qquad (3.5)$$

Then, inserting Eqs. (2.5) in Eq.(3.5), one finds the "cluster" mean-field approximations $E^{cmf}([l_i]; v)$ for eigenenergies,

$$E^{cmf}([l_i]; v) = \tilde{C} + a(v \mp J)c(2r) - 2|g|(J \mp v)s(2r)\sqrt{\Phi((\mp J + v)c(2r))} \qquad (3.6a)$$

where r is determined from the equation

$$\frac{a}{|g|}s(2r) = \pm 2c(2r)\sqrt{\Phi(\mp Jc(2r))} + \frac{J[s(2r)]^2\Phi'(\mp Jc(2r))}{\sqrt{\Phi(\mp Jc(2r))}}, \quad \Phi'(\mp Jc(2r)) = \frac{\partial\Phi(x)}{\partial x}\Big|_{x=\mp Jc(2r)}$$

$$(3.6b)$$

Let us make some remarks concerning results obtained.

Remark 1. As is seen from Eq. (3.3), its general structure coincides with the energy formula given by Eq. (2.16), and spectral functions $|S_{fv}(J;g,r)S_{f+1v}(J;g,r)|\sqrt{\Psi(l_0+1+f)} = E_f^{\Phi}(r;J;v)$ are nonlinear in the discrete variable v labeling energy levels within $L([l_i])$ that provides a non-equdistant character of energy spectra within fixed subspaces $L([l_i])$ at $d([l_i]) > 3$. Besides, due to the presence of square roots in Eqs. (3.3), (3.6a) different eigenfrequencies $\omega_v \equiv E([l_i];v)/\hbar$ are incommensurable: $m\omega_{v_1} \neq n\omega_{v_2}$ that is an indicator of complex dynamics manifesting in such phenomena as collapses-revivals of the Rabi oscillations [2,8] and singular and pre-chaotic dynamic regimes in phase spaces of models [13,22,23]. Evidently, it is hardly possible to obtain these features of models by using GCS related to uncoupled subsystems (cf. [4,8,18] and Section 5 of the present paper).

Remark 2. In the compact ($su(2)$) case the r.h.s. of Eq. (3.4) is a polynomial of the degree $2J+1$, and, in general, Eq. (3.4) may have $2J+1$ different roots r_i corresponding to $2J+1$ different stationary values of the energy functional $\mathcal{H}([l_i];v;\xi)$. Therefore, one may assume that it is possible to get more simple expressions for certain $E([l_i];v)$ using $E([l_i];0)$ from Eq. (3.3b) with different real roots r_i of Eq. (3.4); specifically, it is the case for dimensions $d([l_i]) = 2$ when Eqs. (3.3)-(3.4) give exact results. However, using the well-known expressions for overlap integrals of $SU(2)$ GCS [16], one can show that in general only two $SU(2)$ GCS with different real roots r_i may be mutually orthogonal.

Remark 3. Obviously, Eq. (3.3) generalizes Eq. (2.7a) for the (quasi)equidistant approximation abovementioned. Indeed, when replacing the functions $\Phi(\mp J + f)$ by their certain (and the same for all labels v) "average" values, series in (3.3), (3.4) are summed up, and Eq. (3.3) is reduced to Eq. (2.7a); Taylor series expansions of functions $\sqrt{\Phi(\mp J + f)}$ provide perturbative corrections related to higher degrees of the an-harmonicity of Hamiltonians (2.2). At the same time Eqs. (3.5)-(3.6) yield an intermediate (related to a more fine "averaging" procedure (3.5)) approximation retaining the main characteristic feature of Eq. (3.3) (a non-equdistant character of energy spectra within fixed subspaces $L([l_i])$) but being simpler in its form that is important from the practical point of view. Besides, Eqs. (3.4) and (3.6b) are simplified in the resonance case when $a = 0$.

Remark 4. In fact, solving Eqs. (3.1b) one can get a whole series of competitive potential solutions (corresponding to different roots r_i and v) which may approximate exact ones with a good accuracy in particular parts of energy spectra. (This situation resembles that occuring in the stationary phase calculations of the path integral approach when one needs to take into account contributions of several classical trajectories [24,25].) A final selection of the most adequate value r_0 may be made with the help of a "quality criterion" of QCAs on subspaces $L([l_i])$. For example, one can estimate an accuracy of QCAs obtained by means of the "energy error" functionals [10]

$$\delta_{[l_i]}^p(H, H^{cq/cmf}) = |Tr_{[l_i]}(H - H^{cq/cmf})^p|/|Tr_{[l_i]}(H)^p|, \quad p = 1, 2 \qquad (3.7)$$

giving "energy-trace" proximity meausures of the exact Hamiltonians (2.4) and their QCAs

$$H^{cq}(\{Y_i\};\xi_0) = \sum_{[l_i],v} E^{cq}([l_i];v)|[l_i];v;\xi_0\rangle\langle\xi_0;v;[l_i]| = \tilde{C} + S_Y(\xi_0)^+\tilde{h}_f(Y_0;\xi_0)S_Y(\xi_0), \qquad (3.8a)$$

$$H^{cmf}(\{Y_i\};\xi_0) = \sum_{[l_i],v} E^{cmf}([l_i];v)|[l_i];v;\xi_0\rangle\langle\xi_0;v;[l_i]| =$$

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$$\tilde{C} + S_Y(\xi_0)^+[a(Y_0)c(2r) \pm 2|g|Y_0 s(2r)[\phi_{n-2}((Y_0)c(2r))]^{1/2}]S_Y(\xi_0) \tag{3.8b}$$

on subspaces $L([l_i])$; $Tr_{[l_i]}A = \sum_v \langle v; [l_i]|A|[l_i]; v\rangle$. Furthermore, functionals (3.7) may be used in alternative "minimization schemes" of determining the paprameter r_0.

4 Variational quasiclassical dynamics of $SL(2)$-clusters and time evolution of uncoupled variables

The energy functionals (3.2) and their mean-field approximations (3.5) may be also used for a quasiclassical analysis of time evolution of cluster dynamical variables related to the $sl(2)$ generators Y_i (cf. [5]). As is known, when Hamiltonians (2.2) are linear in $sl(2)$ generators, quasiclassical dynamics is isomorphic to the exact quantum one [14-16] and is described by the classical Hamiltonian equations [5,14,16]

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p}, \qquad \dot{p} = -\frac{\partial \mathcal{H}}{\partial q}, \qquad \mathcal{H} = \langle z(t); [l_i]|H|[l_i]; z(t)\rangle \tag{4.1a}$$

for "motion" of the canonical parameters p, q of the $SL(2)$ GCS $|[l_i]; z(t)\rangle = \exp(-z(t)Y_+ + z(t)^*Y_-)|[l_i]\rangle$ as trial functions in the time-dependent Hartree-Fock variational scheme [17] with the Lagrangian $\mathcal{L} = \langle z(t); [l_i]|(i\partial/\partial t - H)|[l_i]; z(t)\rangle$; $q = \theta$, $p = \langle z(t); [l_i]|Y_0|[l_i]; z(t)\rangle = \mp Jc(2r)$, $z = r\exp(-i\theta)$. An equivalent formulation can be given in $\mathbf{Y} = (Y_1, Y_2, Y_0)$ space using $sl(2)$ vector Euler-Lagrange equations [5],

$$\dot{\mathbf{y}} = \frac{1}{2}\bigtriangledown\mathcal{H} \times \bigtriangledown\mathcal{C}, \quad \mathbf{y} = (y_1, y_2, y_0), \ y_i = \langle z(t); [l_i]|Y_i|[l_i]; z(t)\rangle, \quad \bigtriangledown = (\partial/\partial y_1, \partial/\partial y_2, \partial/\partial y_0),$$

$$\mathcal{C} = \pm y_0^2 + y_1^2 + y_2^2 = \pm J^2, \ y_\pm = y_1 \pm iy_2, \ \mathbf{A} \times \mathbf{B} = (A_2 B_0 - A_0 B_2, A_0 B_1 - A_1 B_0, A_1 B_2 - A_2 B_1) \tag{4.1b}$$

which yield linear quasiclassical Bloch-type equations for $sl(2)$ linear Hamiltonians [19].

In the general case of nonlinear Hamiltonians (2.2) Eqs. (4.1) with $\mathcal{H}$ given by Eqs. (3.2) and (3.5) at $\xi = z^*, v = 0$ also describe a quasiclassical $SL(2)$ "cluster" dynamics of models under study which, however, is not isomorphic to the exact quantum one [5]. Besides, Eqs. (4.1) obtained with the help of GCS $|[l_i]; z(t)\rangle$ describe dynamics of $SL(2)$ clusters within each subspace $L([l_i])$ separately. Specifically, Eqs. (4.1a) determine a "$sl(2)$ linearized" QA $U^l_{H;[l_i]}(t) \propto P_{[l_i]}\exp(-z(t)Y_+ + z(t)^*Y_-)$ of $L([l_i])$-restricted evolution operators $U_{H;[l_i]}(t) = P_{[l_i]}U_H(t)$ when initial wave functions $|\psi_0\rangle$ are equal to $|[l_i]\rangle$ (cf. [19]); in a sense, this approximation is equivalent to that obtained by substitutions in Eq. (2.2b) time-dependent coupling functions $g(t)$ (compatible with solutions of Eqs. (4.1a)) instead of $g(Y_0)$ (cf. [4,23]). However, for general initial wave functions $|\psi_0\rangle \in L(H)$ it is necessary to generalize these equations, e.g., by using GCS $\exp(-z(t)Y_+ + z(t)^*Y_-)|\psi_0\rangle$. Without dwelling on a detailed analysis of this topic we write down examples of Eqs. (4.1) when appropriate explicit expressions for $\mathcal{H}$ are obtained from Eqs. (3.3b) and (3.6a) at $v = 0$ by means of the substitutions

$$2|g| \to ge^{-iq} + g^*e^{iq} = 2[\Re\{g\}\cos q + \Im\{g\}\sin q], \ 2i\Im\{g\} = g - g^*, \ 2\Re\{g\} = g + g^*,$$

$$\mp Jc(2r) = y_0 = p, \quad \cos q = \frac{-y_1}{\sqrt{\pm(J^2 - y_0^2)}}, \quad \sin q = \frac{y_2}{\sqrt{\pm(J^2 - y_0^2)}} \tag{4.2}$$

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where the first line is taken from the substitution $\xi_0 = rg/|g| \to z^* = r\exp(iq)$ in Eqs.(3.3) and the second one is a direct consequence of Eqs. (2.5).

Then, from Eqs. (3.3b), (4.1b) and (4.2) one gets essentially nonlinear quasiclassical Bloch-type equations

$$\dot{\mathbf{y}} = \frac{1}{2}\nabla\mathcal{H}^{cq} \times \nabla\mathcal{C}, \quad \nabla\mathcal{H}^{cq} = (2\Re\{g\}\Theta(y_0), -2\Im\{g\}\Theta(y_0), a + 2[\Re\{g\}y_1 - \Im\{g\}y_2]\partial\Theta(y_0)/\partial y_0),$$

$$\Theta(y_0) = \left(\frac{J \mp y_0}{2J}\right)^{\pm 2J - 1} \sum_{f \geq 0} \frac{(2J)^{-1}\sqrt{\Phi(\mp J + f)}[y_0 \pm J]^f}{f!(f+1)!N^2(J, f+1)[J \mp y_0]^f}, \quad \nabla\mathcal{C} = 2(y_1, y_2, \pm y_0) \quad (4.3)$$

At the same time, using substitutions (4.2) in Eqs. (3.6a) and inserting them in Eqs. (4.1) one finds in the mean-field approximation (3.5), respectively, canonical Hamiltonians equations

$$\dot{q} = a \mp [\Re\{g\}\cos q + \Im\{g\}\sin q][\pm(J^2 - p^2)\Phi(p)]^{-1/2}\partial[(J^2 - p^2)\Phi(p)]/\partial p,$$

$$\dot{p} = [-\Re\{g\}\sin q + \Im\{g\}\cos q]\sqrt{\pm(J^2 - p^2)\Phi(p)} \quad (4.4a)$$

and more simple in comparison with Eqs. (4.3) nonlinear Bloch-type equations obtained from Eqs. (4.3) by the substitution

$$\nabla\mathcal{H}^{cq} \to \nabla\mathcal{H}^{cmf} =$$

$$(2\Re\{g\}[\Phi(y_0)]^{1/2}, -2\Im\{g\}[\Phi(y_0)]^{1/2}, a + [\Re\{g\}y_1 - \Im\{g\}y_2)][\Phi(y_0)]^{-1/2}\partial\Phi(y_0)/\partial y_0) \quad (4.4b)$$

Note that these latter Bloch-type equations are equivalent to those obtained in [10] in terms of variables $v_i(t) = <V_i(t)>$ and solved in terms of hyperelliptic functions.

So, Eqs. (4.1)-(4.2) and their specifications (4.3)-(4.4) yield a tool for examining quasiclassical dynamics of $SL(2)$ clusters within subspaces $L([l_i])$. However, they are not suitable for such an analysis at arbitrary initial conditions or for time-evolution of uncoupled (characterizing single subsystems) dynamical variables that is often necessary in practice. At the same time Eqs. (3.3) and (3.6) enable to obtain appropriate QAs

$$U_H^{cq}(t) = \sum_{[l_i],v} S_Y(\xi_0)^\dagger \exp(\frac{-itE([l_i];v)}{\hbar}) \,|[l_i];v\rangle\langle v;[l_i]|\, S_Y(\xi_0) =$$

$$\sum_{[l_i]}\sum_{v \geq 0} \exp(\frac{-itE([l_i];v)}{\hbar}) \sum_{f \geq 0}\sum_{f' \geq 0} S_{fv}(J;g,r)S_{f'v}^*(J;g,r)|[l_i];f\rangle\langle f';[l_i]| \quad (4.5)$$

of evolution operators $U_H(t)$ when eigenenergies $E([l_i];v)$ are given by Eqs. (3.3a)-(3.4) or Eqs. (3.6). Evidently, $L([l_i])$-restrictions $P_{[l_i]}U_H^{cq}(t)$ of such evolution operators (4.5) are distinguished from evolution operators $U_{H;[l_i]}^l(t)$ associated with solutions of Eqs. (4.1a).

Substitutions of Eqs. (3.3a)-(3.4) or Eqs. (3.6) in Eqs. (4.5) enable us to calculate appropriate QAs for time-dependences

$$< F(t) >= \mathrm{Tr}[U_H^{cq}(t)\,\rho\,U_H^{cq}(t)\,F] \quad (4.6)$$

of any dynamical variables F where ρ is a density operator for an initial quantum state. For example, inserting in Eq. (4.6) ordered exponentials of coupled (Y_i) or uncoupled (original) dynamical variables one may get (after an appropriate Fourier transformation) formulas describing dynamics of different (associated with GCS of both $SL(2)$ and dynamic symmetry

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groups of subsystems) types of Q-,P- and Wigner quasiprobabilty functions which are widely used for visualizing features of systems under study [14,26]. Note also that, due to Eqs. (2.5), the first line in Eq. (4.5) is more suitable for using Eq. (4.6) with $F = F(\{Y_i\})$ whereas the second one is more relevant for calculations with F depending on uncoupled dynamical variables.

5   Applications to a class of quantum-optical models

In this Section we manifest a physical meaning of general results above on ceveral concrete models which are widely applied in quantum optics, laser physics and quantum electronics [3,4,8,18,19,25]. Specifically, as was shown in [11], a natural area of applications of the $sl_{pd}(2)$ formalism is provided by quantum models with Hamiltonians

$$H_1/\hbar = \sum_{i=1}^{2} \omega_i a_i^+ a_i + g'(a_1^+)^m (a_2)^n + g'^*(a_1)^m (a_2^+)^n, \; n \leq m, \qquad (5.1a)$$

$$H_2/\hbar = \sum_{i=1}^{m} \omega_i a_i^+ a_i + \omega_0 a_0^+ a_0 + g'(a_1^+ ... a_m^+)(a_0)^n + g'^*(a_1 ... a_m)(a_0^+)^n, \; n \leq m, \qquad (5.1b)$$

$$H_3/\hbar = \omega_1 a_1^+ a_1 + \sum_{i=1}^{N}[\sigma_0(i)\epsilon/2 + g'\sigma_+(i)(a_1)^n + g'^*\sigma_-(i)(a_1^+)^n] \qquad (5.1c)$$

where g' are coupling constants, a_i, a_i^+ are boson operators describing field modes with frequencies ω_i, $\sigma_\alpha(i)$ are Pauli matrices, ϵ is an energy difference of two level atoms and non-quadratic parts of H_i describe different multiphoton processes of scattering and frequency conversion (Eqs. (5.1a,b)) as well as the matter-radiation interactions in n-photon point-like Dicke models in rotating wave approximation (Eqs. (5.1c)). Note that in applications, one considers, as a rule, models (5.1) with $n = 0, 1$ that correspond, respectively, to semiclassical or completely quantum versions of models under study [3,4,8,18,23,25].

Appropriate Hilbert spaces $L(H_i)$ are multimode Fock spaces $L_F = Span\{|\{n_i\}\rangle = \prod_i[n_i!]^{-1/2}(a_i^+)^{n_i}|0>\}$ for models (5.1a)-(5.1b) whereas for models (5.1c) $L(H_3)$ are direct products of single-mode Fock spaces L_F and "atom" spaces $L_a = Span\{|j, \mu; \{j_{int}\} >\rangle\}$ where $|j, \mu; \{j_{int}\} >$ are the basis vectors of irreducible representations of the "atom" group $SU(2)^a$ (with generators $\Sigma_\alpha = \sum_{i=1}^{N} \sigma_\alpha(i)$) which are obtained from one-atom basis states $|\pm > (i)$ with the help of the generalized Wigner coefficients and $\{j_{int}\}$ are sets of the $SU(2)^a$ intermediate angular momenta labeling basis vectors of the irreducible representations of the symmetric group S_N and being integrals of motion [11].

Hamiltonians (5.1) are expressed in the form (1.1)-(1.2) with the help of introducing $sl_{pd}(2)$ dynamic variables $V_0, V_+, V_- = (V_+)^\dagger$ and integrals of motion R_j via a generalized Jordan-Schwinger mapping [10] given for H_1, H_2, H_3 respectively as follows [11]:

$$V_0 = \frac{1}{m+n}(a_1^+ a_1 - a_2^+ a_2), \; V_+ = (a_1^+)^m (a_2)^n, \quad R_1 = \frac{1}{m+n}(na_1^+ a_1 + ma_2^+ a_2), \qquad (5.2a)$$

$$V_0 = \frac{1}{m+n}(\sum_{i=1}^{m} a_i^+ a_i - a_0^+ a_0), \qquad V_+ = a_1^+ ... a_m^+ (a_0)^n,$$

$$R_k = \frac{1}{m+n}(a_j^+ a_j - a_{j+1}^+ a_{j+1}), \; k = 1, \ldots, m-1, \quad R_m = \frac{1}{m+n}(n \sum_{i=1}^{m} a_i^+ a_i + m a_0^+ a_0), \quad (5.2b)$$

$$V_0 = \frac{1}{2} \sum_{i=1}^{N} \sigma_0(i), \quad V_+ = \sum_{i=1}^{N} \sigma_+(i)(a_1)^n, \quad R_1 = \frac{n}{2} \sum_{i=1}^{N} \sigma_0(i) + a_1^+ a_1 \quad (5.2c)$$

The structure polynomials $\Psi(V_0)$ are determined with the help of Eqs. (5.2) (and defining relations for $a(i), a^+(i), \sigma_\alpha(i))$ from Eq. (1.3) which is valid for all $L(H_i)$. Then for H_1, H_2, H_3 one finds, respectively,

$$\Psi(V_0) = (mV_0 + R_1)^{(m)}(R_1 - nV_0 + n)^{(n)}, \quad (5.3a)$$

$$\Psi(V_0) = [R_m - nV_0 + n]^{(n)}([R_m - (m+n)\sum_{i=1}^{m-1} iR_i]/m + V_0) N_1 N_2 \ldots N_{m-1},$$

$$N_k = \frac{1}{m}[R_m - (m+n)\sum_{i=1}^{m-1} iR_i] + V_0 + (m+n)\sum_{i=k}^{m-1} R_i, \; k = 1, \ldots, m-1, \quad (5.3b)$$

$$\Psi(V_0) = [C_2(2) - V_0^{(2)}][R_1 - nV_0 + n]^{(n)} \quad (5.3c)$$

where $C_2(2) = \Sigma_+ \Sigma_- + (\Sigma_0/2)^{(2)}$ is the Casimir operator of the "atom" $su(2)$ algebra and $[C_2(2), V_\alpha] = 0, A^{(B)} = A(A-1)\ldots(A-B+1)$.

The subspaces $L([l_i])$ in Eq. (1.4) are generated by the lowest vectors $|[l_i] >$ which are given for different H_i as follows
H_1 :

$$|[l_i] >= |\{n_1 = \kappa, n_2 = s\}\rangle, \quad l_0 = \frac{1}{m+n}(\kappa - s), \quad l_1 = \frac{1}{m+n}(n\kappa + ms),$$

$$R_j|[l_i] >= l_j|[l_i] >, \; \kappa = 0, 1, \ldots, m-1, \; s = 0, 1, \ldots, \quad (5.4a)$$

H_2 :

$$|[l_i] >= |\{n_1 = \kappa_1, n_2 = \kappa_2, \ldots, n_m = \kappa_m, n_0 = s\}\rangle, \qquad \prod_i^m \kappa_i = 0,$$

$$l_0 = \frac{1}{m+n}(\sum_{i=1}^{m}\kappa_i - s), \; l_m = \frac{1}{m+n}(n\sum_{i=1}^{m}\kappa_i + ms), \; l_k = \frac{(\kappa_j - \kappa_{j+1})}{m+n}, \; k = 1, \ldots, m-1,$$

$$R_j|[l_i] >= l_j|[l_i] >, \; \kappa_i = 0, 1, \ldots, \; s = 0, 1, \ldots, \quad (5.4b)$$

H_3 :

$$|[l_i] >= |\{n_1 = \kappa\}\rangle|j, \mu = -j; \{j_{int}\} >, \quad l_0 = -j, \quad l_1 = \kappa - nj,$$

$$R_j|[l_i] >= l_j|[l_i] >, \quad \kappa = 0, 1, \ldots, \; 0 \le j \le N/2, \quad (5.4c)$$

where $|\{n_i\}\rangle$ are standard Fock states and $|j, -j; \{j_{int}\} >$ are lowest vectors of irreducible representations of the "atom" group $SU(2)^a$. From Eqs. (5.3)-(5.4) it follows that we have compact versions of algebras $sl_{pd}(2)$ in all cases except for models (5.1a,b) with $n = 0$.

Eqs. (5.3)-(5.4) yield requisites for specifications of general results of Sections 3 and 4. However, for the sake of simplicity of our exposition, we restrict ourselves by considering certain simple examples which elucidate main features of new QAs and, simultaneously, will provide a base for further investigations of the most spread in applications models and physically important cases ($n \le 1, m \le 3$ in Eqs. (5.1a)-(5.1b) and $n = 1$ in Eq. (5.1c)).

Example 1. "Cluster" *mean-field energy spectra in models (5.1) with* $n = 1, m = 2, 3$. Inserting Eqs. (5.3) in compact ($su(2)$) versions of Eq. (2.1a) and using Eqs. (5.4) one finds for the polynomials $\Phi(Y_0)$ from Eqs. (2.1a) the following expressions

$$H_1 : \quad \Phi^m(Y_0) = \frac{(mY_0 + mJ + m + \kappa)^{(m)}}{(J + 1 + Y_0)}, \qquad J = s/2, \qquad X^{(m)} = X(X-1)\ldots, \quad (5.5a)$$

$$H_2 : \qquad \Phi^m(Y_0) = \prod_{i=1}^{m}{}'(Y_0 + J + \kappa_i + 1) = \prod_{i=1}^{m}{}'(Y_0 + \frac{s}{2} + \kappa_i + 1), \qquad \prod_{i=1}^{m}\kappa_i = 0, \quad (5.5b)$$

$$H_3 : \quad \Phi(Y_0) = \max(\kappa, 2j) - J - Y_0 = \begin{cases} (\kappa - j - Y_0), & \kappa \ge 2j, \\ (2j - \frac{\kappa}{2} - Y_0), & \kappa \le 2j \end{cases} , \; J = \min(j, \frac{\kappa}{2}) \quad (5.5c)$$

where it is also taken into account that $d([l_i]) = 2J + 1$ and simultaneously $d([l_i]) = s + 1$ for $H_i, i = 1, 2$ and $d([l_i]) = \min(2j, \kappa) + 1$ for H_3; besides, the numerator in Eq.(5.5a) always contains the factor $(J + 1 + Y_0)$ due to the definition of the symbolic powers $X^{(m)}$ and $\prod_{i=1}^{m}{}'$ in Eq. (5.5b) means that in the product the term with $\kappa_i = 0$ is omitted. Then, inserting Eqs.(5.5) in Eqs. (3.3),(3.4),(3.6) and Eqs. (4.3)-(4.5) and using also Eqs. (5.4) one can obtain appropriate specifications of QAs above for energy spectra, Bloch-type dynamical equations and evolutions operators and to examine their features depending on characteristic parameters of models under study.

However, postponing such a detailed analysis for further publications, we only write down appropriate specifications of Eqs. (3.6a) in the resonace cases ($a = 0$ in Eq. (2.2)) and at $m = 2$ in Eqs. (5.5a,b) when Eqs. (3.6b) are solved analytically yielding

$$\cos 2r = c(s, \kappa) = \frac{1}{3}\left(\frac{2\kappa + 1}{s} + 1 - 2\sqrt{1 + (\frac{2\kappa + 1}{2s})(\frac{2\kappa + 1}{2s} + 1)}\right)$$

$$\approx \frac{1}{3}(\frac{2\kappa + 1}{2s} - 1)(\text{for } s > 2\kappa + 1) \qquad (5.6a)$$

$$\cos 2r = c(s, \kappa) = \frac{1}{3}\left(\frac{2\kappa + 2}{s} + 1 - 2\sqrt{1 + (\frac{\kappa + 1}{s})(\frac{\kappa + 1}{s} + 1)}\right)$$

$$\approx \frac{1}{3}(\frac{\kappa + 1}{s} - 1)(\text{for } s > \kappa + 1) \qquad (5.6b)$$

$$\cos 2r = c(j, \kappa) = \frac{1}{3}\left(1 - 2\mu(\kappa, j) + 2\sqrt{1 - \mu(\kappa, j) + (\mu(\kappa, j))^2}\right), \quad \mu(\kappa, j) = \frac{\max(\kappa, 2j)}{\min(\kappa, 2j)} \qquad (5.6c)$$

for H_1, H_2, H_3 respectively. Then, with the help of Eqs. (5.5)-(5.6) eigenenergies $E^{cmf}([l_i]; v)$ in the "cluster" mean-field approximation (3.5) are given as follows,
H_1:

$$E^{cmf}([l_i]; v)/\hbar - \tilde{C} = E^{cmf}(\kappa, s; v)/\hbar - (\kappa + 2s)\omega_1 =$$

$$-|g'|(s - 2v)\sin 2r\sqrt{2((-s + 2v)\cos 2r + s + 2\kappa + 1)} =$$

$$-|g'|(s - 2v)\sqrt{(1 - [c(s, \kappa)]^2)2((-s + 2v)c(s, \kappa) + s + 2\kappa + 1)}, \quad \omega_2 = 2\omega_1 \qquad (5.7a)$$

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H_2:

$$E^{cmf}([l_i];v)/\hbar - \tilde{C} = E^{cmf}(\kappa,s;v)/\hbar - \kappa_1\omega_1 + \kappa_2\omega_2 + s(\omega_1+\omega_2) =$$

$$-|g'|(s-2v)\sin 2r\sqrt{(-\frac{s}{2}+v)c(s,\kappa)+\frac{s}{2}+\kappa+1)} =$$

$$-|g'|(s-2v)\sqrt{(1-[c(s,\kappa)]^2)((-\frac{s}{2}+v)c(s,\kappa)+\frac{s}{2}+\kappa+1)},$$

$$\kappa_1\kappa_2 = 0, \quad \kappa = \max(\kappa_1,\kappa_2), \quad \omega_0 = \omega_1+\omega_2 \qquad (5.7b)$$

H_3:

$$E^{cmf}([l_i];v)/\hbar - \tilde{C} = E^{cmf}(\kappa,j;v)/\hbar - (\kappa-j)\omega_1 =$$

$$-|g'|(s-2v)\sin 2r\sqrt{\max(\kappa,2j)-J-(-J+v)\cos 2r} =$$

$$-|g'|(s-2v)\sqrt{(1-[c(s,\kappa)]^2)(\max(\kappa,2j)-J-(-J+v)c(s,\kappa))}, \quad \epsilon = \omega_1, \ J = \min(j,\frac{\kappa}{2})$$
$$(5.7c)$$

Evidently, Eqs. (5.7) manifest explicitly an essentially nonlinear dependence of energy levels $E^{cmf}([l_i];v)$ on their both $su(2)$-invariant (κ,s,j) and non-invariant (v) labels unlike standard QAs obtained by means of using in Eqs. (3.1) GCS associated with dynamic symmetry algebras of subsystems. Indeed, using in variational schemes (3.1)-(3.2) Glauber's CS $\prod_i \mathcal{D}(\alpha_i)|\{n_i\}\rangle$, $\mathcal{D}(\alpha_i) = \exp(\alpha_i a_i^+ - \alpha_i^* a_i)$ for models (5.1) and, additionally, "atomic" $SU(2)^a$ GCS $\exp(\xi\Sigma_+ - \xi^*\Sigma_-)|j,\mu;\{j_{int}\}>$ for models (5.1c) as trial functions , one finds the following analogs of Eqs. (5.7) for such simplest QAs

$$H_1: \qquad E^{smf}(n_1,n_2)/\hbar = \omega_1(n_1+2n_2) + \Lambda_1(\omega_1,|g'|), \qquad \omega_2 = 2\omega_1 \qquad (5.8a)$$

$$H_2: \ E^{smf}(n_1,n_2,n_0)/\hbar = \omega_1(n_1+n_0) + \omega_2(n_2+n_0) + \Lambda_2(\omega_1,\omega_2,|g'|), \ \omega_0 = \omega_2+\omega_1 \quad (5.8b)$$

$$H_3: \qquad E^{smf}(n_1,\mu)/\hbar = \omega_1 n_1 + \mu\Omega(\omega_1,|g'|) + \Lambda_3(\omega_1,|g'|), \qquad \epsilon = \omega_1 \qquad (5.8c)$$

where $\Lambda_i(\ldots)$ are constant (for whole $L(H)$) energy shifts and Ω is an efficient frequency. Evidently, energy levels (5.8) depend linearly on G_{inv}-noninvariant labels n_i arranged on multidimensional lattices that provides multiperiodic dynamical regimes. Other ordinary QAs [4,8,25], e.g., obtained with the help of GCS of partially coupled subsystems, lead to similar results (as it is seen, in fact, from comparisons of Eqs.(5.8b) and (5.8c)).

Example 2. "Cluster" mean-field energy spectra in models (5.1a,b) with $n = 0, m = 3$. In this case models under study yield so-called parametric approximations for models of the first example with $m = 3$. Besides, we have noncompact versions of $sl_{pd}(2)$ algebras because all subspaces $L([l_i])$ are infinite-dimensional. This, in turn, causes an ambiguity of determining the parameter J in the generalized Holstein-Primakoff mappings (2.1) on subspaces $L([l_i])$ because a polynomial character of $\Phi(Y_0) = \Psi(Y_0 + l_0 - J + 1)/(J + Y_0)(-J + 1 + Y_0)$ is provided by two values of J on each subspace $L([l_i])$ that requires to add a choice procedure of J to Eqs. (3.1). However, we restrict ourselves by writing down analogs of Eqs. (5.5a,b),

$$H_1: \qquad \Phi^3(Y_0) = \frac{(3Y_0-3J+3+\kappa)^{(3)}}{(-J+1+Y_0)(J+Y_0)} = 27(Y_0 + \lambda_1(\kappa,J_\kappa)), \qquad \kappa = 0,1,2, \quad (5.9a)$$

$$H_2: \qquad \Phi^3(Y_0) = \frac{\prod_{i=1}^{3}(Y_0 - J + 1 + \kappa_i)}{(-J + 1 + Y_0)(J + Y_0)} = Y_0 + \lambda_2(\{\kappa_i\}, J_{\kappa_i}), \qquad \prod_{i=1}^{3} \kappa_i = 0, \qquad (5.9b)$$

which, nevertheless, manifest differences of parametric QAs from those given by Eqs. (5.5) due to linear and quadratic forms of $\Phi^3(Y_0)$ in these cases (constants $\lambda_i(\ldots, J_{...})$ are easily determined for chosen values $\{\kappa_i\}, J_{\kappa_i}$). A more detailed analysis of such comparisons will be given elsewhere.

6   Conclusion

So, we have obtained new approximations for energy spectra and evolution operators as well as nonlinear Bloch-type dynamic equations for models (1.1) (and (5.1)) by means of using the mapping (2.1) and standard variational schemes [17,5] with the $SL(2)$ GCS as trial functions. They may be called as "cluster" (or correlated) QAs owing to taking into account strong quantum correlations between interacting subsystems. These approximations may be used to calculate in models of the (5.1) type time evolution of different quantum-statistical characteristics and quasidistributions (cf. [8,14]) and to find bifurcation sets and solutions of nonlinear Hamiltonian flows determined by Eqs. (4.1) and (4.5) (cf. [5,23]). In this way we hope to reveal in these models new cooperative phenomena and dynamical regimes (due to quantum correlations between subsystems) by analogy with those found in [9,18,22,23,27] and many other papers by using standard QAs; herewith different QAs above are expected to elucidate the role of such correlations depending on a choice of initial quantum states and paprameters of models under study (cf. [22,23]).

However, from the practical point of view for this aim it is desirable to modify and to simplify Eqs. (3.3) and (4.5) by using different properties of the hypergeometric functions $_2F_1(a, b; c; x)$, including their integral representations and asymptotic expansions [20,28]. (Specifically, in such a way one can express spectral functions $E_f^{\Phi}(r; J; v)$ in terms of the hypergeometric functions $_4F_3(...; 1)$ which are proportional to the $sl(2)$ Racah coefficients [12].) Along this line it is also of importance to get estimations of accuracy of QAs obtained and of their efficiency in comparison with other approximations (e.g., considerd in [8,9,10,18]). One way to do such estimations is in comparisons of these QAs with appropriate computer calculations (cf. [18,27]) and another one is connected with using the "energy-trace" proximity meausures (3.7).

Another line of further investigations concerns developments of mathematical aspects of the work. Indeed, results of Sections 3,4 correspond to picking out "smooth" $sl(2)$ factors $S_Y(\xi_0) = \exp(\xi_0 Y_+ - \xi_0^* Y_-)$ in exact (generally, not "smooth") diagonalizing operators S determined by Eqs. (2.9), (2.12) and, when using Eqs. (4.1a), in evolution operators $U_H(t)$ determined by exact evolution equations given in [11]; besides, Eqs. (4.5) yield another type of QA for evolution operators $U_H(t)$. All these QAs can be used as initial approximations in iterative schemes of constructing exact solutions which are similar to those developed to examine nonlinear problems of classical mechanics and optics [29] or in search of suitable multi-parametric improvements of variational schemes used by introducing "form-factors" with extra fitting parameters in original trial functions. It is also of interest to develop methods of obtaining simple formulas for exact solutions of tasks under consideration in order to compare with them results of approximations found above. At present one may to point out, at least, three promising ways along this line.

One of them is in simplifications of integral solutions obtained in [11] for both evolution and spectral tasks. The second way, leading to solving singular differential equations, is connected with using two conjugate differential realizations of $sl_{pd}(2)$ generators V_α [10,12]:

$$V_+ = z, \quad V_0 = zd/dz + l_0, \quad V_- = z^{-1}\Psi(zd/dz + l_0), \qquad (6.1a)$$

$$V_- = d/dz, \quad V_0 = zd/dz + l_0, \quad V_+ = \Psi(zd/dz + l_0)(d/dz)^{-1} \qquad (6.1b)$$

which are, in turn, related to realizations of $sl_{pd}(2)$ generators V_α by quadratic forms in $sl(2)$ generators Y_α taken in the coherent-state representations (cf. [30,15]). (In fact, these realizations were used implicitly when obtaining exact integral solutioms [11].) For example, when the structure polynomial $\Psi(x)$ has the third degree (as, e.g., in models (5.1) with $n = 1, m = 2$), the realization (6.1b) reduces original tasks to solutions of the Riccati equations [12]. In this connection one may consider the hypergeometric functions $_2F_1(a, b; c; x)$ determining QAs obtained as specific asymptotics of new classes of special functions determining exact solutions that opens a possibility to use the techniqe of asymptotic expansions [28] for finding latters. Last (but not least!) way is due to interrelationships between $sl_{pd}(2)$ algebras and certain q-deformed algebras mentioned in Section 2 that enables us to use for purposes formulated above techniques of q-deformed algebras and q-special functions, in particular, q-exponents defined with the help of the coherent states map of the paper [21]. Evidently, a progress in solving all these problems will promote to an extension of the orbit type GCS concept [16] and, simultaneously, to a more fine description of "classical" phase spaces associated with dynamic symmetry algebras $sl_{pd}(2)$ (cf. [31]).

The work along these lines is now in progress.

7   Acknowledgements

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